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Influence of Non-Linear Stress-Strain Curve on Elastic Response of Cantilever

Takashi Uchimura[1] and Masayoshi Akiyama[1,*]

[1] Department of Mechanical Engineering, Kyoto Institute of Technology,
Gosho-Kaido-cho, Matsugasaki, Sakyo-ku, 606-8585, Japan

*e-mail : akiyama@mech.kit.ac.jp, Tel & Fax : +81-(0)75-724-7316 (with facsimile)

Abstract

Influence of non-linearity of a stress-strain curve on the elastic response of a cantilever was investigated theoretically and the validity was checked by experiment. Two types of non-linearity were taken into consideration; difference in Young's modulus between the tension and compression sides and the non-linearity of the stress-strain curve. The theory was extended to predict the response of a sheet coated with a thin hard layer. Specimens sectioned from a cold-rolled steel sheet were subjected to tension and compression tests to measure the non-linearity. For the validity check bending tests were carried out.

Keywords: Cantilever, Non-Linear, Stress-Strain curve, Bending Test.

1. Introduction

It is well known that the theory of elasticity well predicts the response of products as long as the material behaves as a linear elastic body [1]. However, there are cases when a material shows a strong non-linearity. Bauschinger's effect [2] is a typical example. For such kind of materials the non-linearity must be taken into consideration in predicting the behaviour of a product. Proper consideration is necessary not only for an elastic response but also for a plastic response [3].

It is a common sense that heavily cold-worked steel shows a high non-linearity [4]. Close investigation was given on the mechanism of this kind of non-linear behaviour by using a multi-scale analysis [5], but in the present work a macroscopic approach is adopted to propose a theory for predicting the non-linear behaviour of a product. Tension and compression tests were carried out to know the initial response of material to predict the response of a cantilever followed by a bending experiment to show the validity.

2. Theory

Theory of bending for a cantilever is based on a theory of linear elasticity and also on an assumption of equality of the Young's moduli on tension and compression sides. In order to deal with the cases for which these assumptions are not applicable the theory must be modified taking into account the behaviour of materials. In the following sections theories are separately derived for a linear elastic body with different elastic moduli on compression and tension sides, for a sheet with non-linear response and for a sheet with a thin hard coated layer on the surface.

2.1 Linear elastic body

Conventional theory of a cantilever consists of two basic equations; equilibrium of axial force and equilibrium of moment in any cross section in the axial direction. These equations are expressed by Eqs. (1) and (2) respectively.

$$\int_{-H}^{H} \sigma w dy = 0 \quad (1)$$

$$\int_{-H}^{H} \sigma w (y-\eta) dy = M \quad (2)$$

where H, w are the half height and width of cantilever, and y is the coordinate value in the thickness direction with origin being placed on the centre plane, and η is the position of neutral plane. σ is the axial stress in the cross section and M is the moment generated by the external force exerting at the top end of cantilever. M is calculated by Eq. (3).

$$M = (L - x) \cdot F \quad (3)$$

where L, x and F are the length of cantilever, axial position in the cantilever from the fixed end and the force exerting at the free end of cantilever.

For a linear elastic body the stress-strain relationship is written by Eq. (4).

$$\sigma = E \cdot \varepsilon \quad (4)$$

where E is the Young's modulus. Normally the values of E on the tension and compression sides are assumed equal and η is zero accordingly. The axial strain ε is calculated by Eq. (5) using ρ that is the radius of curvature of the cantilever in the x-y-plane at axial position x.

$$\varepsilon = \frac{(y-\eta)}{\rho} \quad (5)$$

The position of neutral plane η and the radius of curvature ρ are obtainable by solving these sets of equations. As a result of linearity the value η is zero. The deflection of cantilever at the top end is calculated by integration of the tangent of the curve of cantilever.

2.2 Non-linear body

In the present work the term "non-linearity" is used in two senses; the first one is for a body with a linear stress-strain relationship but with different Young's moduli on the tension and compression sides, and the second one is for a body that has a curved relationship between the stress and the strain.

For a body with different Young's moduli the basic equations of cantilever are different from Eqs. (1) and (2) and are written by Eqs. (6) and (7), and the Eq.(4) is divided into two Eqs. (8) and (9).

$$\int_{-H}^{\eta} \sigma_c w dy + \int_{\eta}^{H} \sigma_t w dy = 0 \quad (6)$$

$$\int_{-H}^{\eta} \sigma_c w (y-\eta) dy + \int_{\eta}^{H} \sigma_t w (y-\eta) dy = M \quad (7)$$

$$\sigma_c = E_c \cdot \varepsilon \qquad (8)$$

$$\sigma_t = E_t \cdot \varepsilon \qquad (9)$$

where σ_c and σ_t are stresses on compression and tension sides and Ec and Et are the Young's moduli on compression and tension sides respectively. Because of the difference in the Young's modulus on tension and compression sides the position η of the neutral plane does not coincides with the position of centre plane, i.e. η is not zero. Solving Eq. (6) by using Eqs. (8) and (9) gives the position η of the neutral plane as is shown by Eq. (10).

$$\eta = \frac{(E_t - E_c)^2}{E_t - E_c} \cdot H \qquad (10)$$

Comparison of responses of cantilevers made of a linear body and a non-linear body with different Young's moduli on tension and compression sides are shown in Fig. 1. For the former the Young's modulus of 200 GPa was assumed and for the latter 200 GPa on tension side and different values were assumed on the compression side; 200 GPa, 180 GPa and 160 GPa. The length and width of cantilever were 130mm and 5mm respectively. Difference in Ec leads to the difference in deflection.

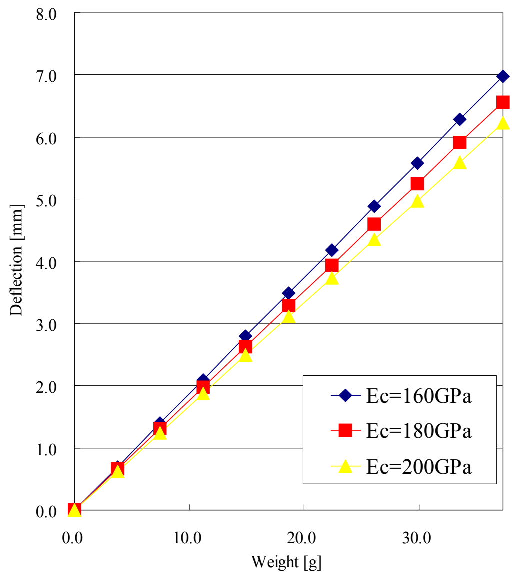


Fig. 1 Comparison of responses of linear and non-linear bodies

For a body of which stress-strain curve is non-linear Eqs. (8) and (9) should be modified according to the response. If the curve can be expressed approximately by a second order polynomial of strain ε the stress-strain relationships are written as follows.

$$\sigma_c = \alpha_c \cdot \varepsilon^2 + \beta_c \cdot \varepsilon \qquad (11)$$

$$\sigma_t = \alpha_t \cdot \varepsilon^2 + \beta_t \cdot \varepsilon \qquad (12)$$

where α_c, β_c, α_t and β_t are coefficients to be calculated from the measured curve.

Numerical examples are illustrated in Fig. 2 for cases with the following values of α_c, β_c, α_t and β_t.

$\alpha_t = -2.8 \times 10^4$

$\beta_t = 208.9$

$\alpha_c = 2.8 \times 10^4$

$\beta_c = 200, 180, 160$

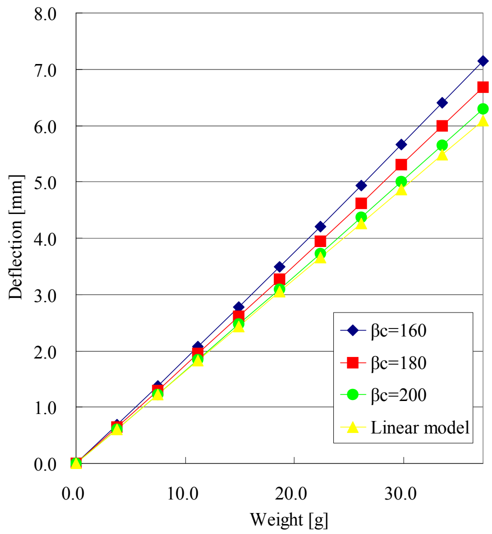


Fig. 2 Influence of non-linear stress-strain curve on deflection of cantilever

2.3 Thin-coated body

For a linear elastic body with a thin hard coated layer on the surface Eqs. (13)-(15) are applicable and some numerical examples are given in Fig. 3.

$$\int_{-H}^{\eta} \sigma_c w dy + \int_{\eta}^{\delta} \sigma_t w dy + \int_{\delta}^{H} \sigma_c w dy = 0 \quad (13)$$

$$\int_{-H}^{\eta} \sigma_c w(y-\eta) dy + \int_{\eta}^{\delta} \sigma_t w(y-\eta) dy + \int_{\delta}^{H} \sigma_{tc} w(y-\eta) dy = M \quad (14)$$

$$\sigma_{tc} = E_{tc} \cdot \varepsilon \quad (15)$$

where δ, σ_{tc} and E_{tc} are the position of the lower surface of the coated layer, the axial stress and the Young's modulus of the thin coated layer.

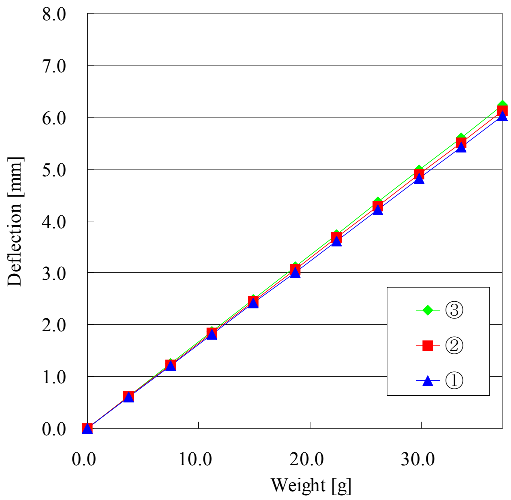


Fig. 3 Influence of thin hard coated layer on deflection of cantilever

3. Measurement of Stress-Strain Curve

Cold rolled steel sheet and the same sheet but with a thin hard coated layer on the surface were subjected to tension and compression tests to measure the stress-strain curve. In the present case the steel sheet was 0.8mm in thickness and the thickness of hard coated layer was 2 μ m. Because the ratio of thickness of the coated layer is 0.125% against the thickness of steel the influence of Young's modulus of the coated layer on the resultant stress-strain curve is negligibly small and it can be allowed to deem the measured curve as that of the steel sheet.

The geometry of specimen for tension and compression tests is given in Fig. 4. The specimens were sectioned from parent sheets shown in Fig. 5 by using a wire cutting technology to leave small residual stress.

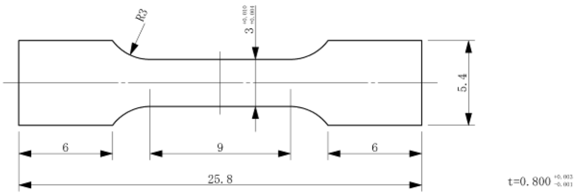
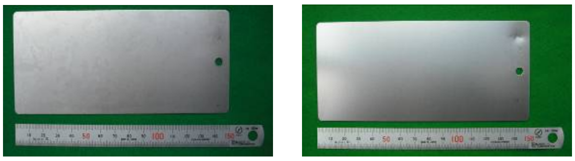


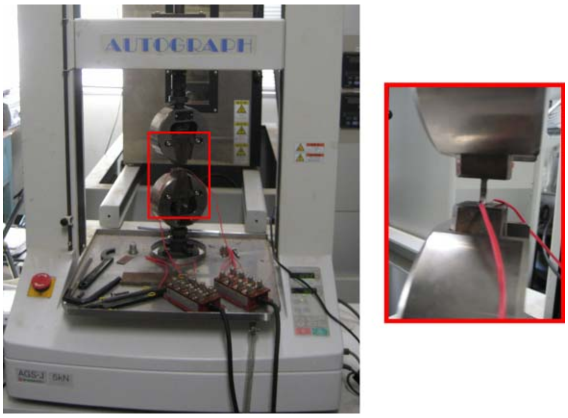
Fig. 4 Geometry of specimen for tension and compression tests



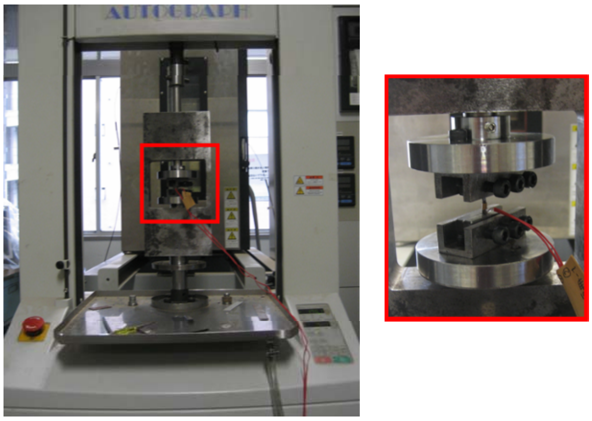
(a) Cold-rolled steel sheet (b) Hard-coated steel sheet

Fig. 5 Views of parent sheets

View of tension and compression tests are shown in Fig. 6. Referring to the result of previous work [6] a pair of strain gauges was placed at the centre in the longitudinal direction of specimen for the precise measurement of strain.



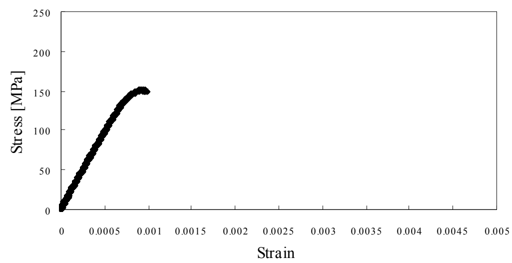
(a) View of tension test



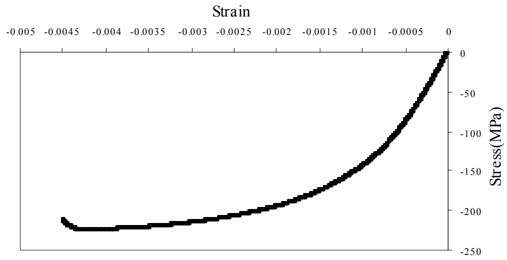
(b) View of compression test

Fig. 6 Views of tension and compression tests

Compression test was carried out until buckling was observed by comparing the signals of two strain gauges. Measured stress-strain curves for the parent steel sheet and those of the coated sheet are shown in Fig. 7 and Fig. 8 respectively.

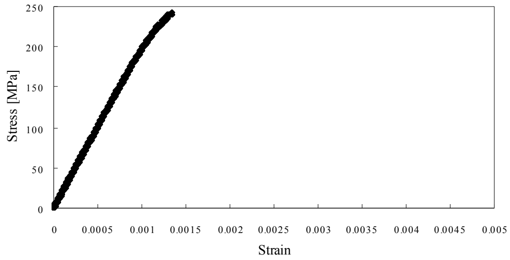


(a) Curve on tension side

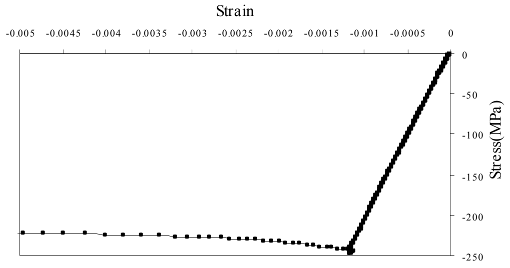


(b) Curve on compression side

Fig. 7 Stress-strain curves of parent steel sheet



(a) Curve on tension side



(b) Curve on compression side

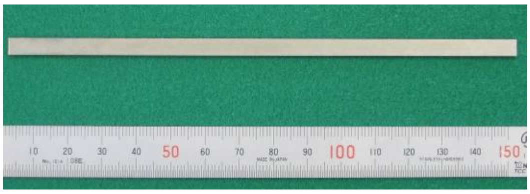
Fig. 8 Stress-strain curves of coated sheet

It is clearly observed that tangents of stress-strain curves on tension and compression sides are different and the tangent on the tension side is larger than that on the compression side. It may be the influence of the Bauschinger's effect because the parent steel sheet is a cold-rolled sheet. If we take the tangent of the curve between the stress levels of 20 and 100 MPa the measured Young's modulus of the parent sheet is 192.1 GPa on tension side and 168.9 GPa on compression side. Similarly those for the coated sheet are 201.0 GPa and 186.4 GPa respectively. It is curious to note that the tangents of stress-strain curves of coated sheet are steeper than those of the parent sheet. This result suggests that there were some influence of heat throughout the operation of coating on the recovery of mechanical property on the cold-rolled parent sheet.

If the stress-strain curve is granted non-linear curve fitting by a second order polynomial by Eqs. (11) and (12) gives an excellent fitting of the stress-strain curve especially for the parent steel sheet. For the coated sheet the stress-strain curve can be deemed straight and the predicted deflection on the assumption of linearity may be give a satisfactory result.

4. Bending Experiment

Bending test was carried out by using a thin cantilevers made from the parent cold-rolled steel sheet and the thin-coated sheet. The geometry of cantilever is shown in Fig. 9. The width and length were 5mm and 130mm respectively. They were also sectioned from the parent sheets by the wire cutting technology and were subjected to bending test.



(a) Parent sheet

(b) CrN-Coated material

Fig. 9 Geometry of specimen for bending test of cantilever

View of bending test is shown in Fig. 10. In the vinyl bag suspended on top of the cantilever a weight of 3.73g was added one by one to measure the deflection on the top end and the similar procedure was taken on the unloading process. The total number of weights added in the bag was 10 and the weight was 37.3g.

Fig. 10 View of bending test

5. Results

Comparison of predicted and measured deflections on the top end is given in Fig. 11. If the same Young's modulus 192.1 GPa is assumed for the parent steel sheet on both of the tension and compression sides the predicted deflection is smaller than the measured value as is indicated in Fig. 11.

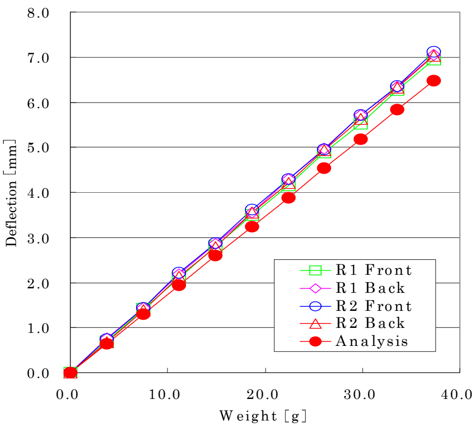


Fig. 11 Comparison of predicted and measured deflections of parent steel sheet

If a smaller Young's modulus is adopted on the compression side the predicted value becomes closer to the measured value. The value of 162.6 GPa on the compression side gives a precise matching.

If a non-linear stress-strain curve is assumed the predicted deflection under the weight of 37.3g is 6.8mm. It is slightly smaller than the measured results but it is closer to the measured values than that of the same Young's modulus.

Measured deflection of a cantilever with a coated layer of CrN was 6.82mm when the weight of 37.3g was applied at the top end. Predicted value of deflection was 6.50mm under the condition such that the Young's moduli were 201.0 GPa on tension side and 186.4 GPa on compression side. The Young's modulus of coated layer of CrN was assumed 300.0 GPa. If the Young's moduli are 201.0 GPa on both sides the deflection is 6.26 mm and the discrepancy is larger, and the non-linearity should be taken into consideration.

6. Conclusions

Influence of non-linear stress-strain curve on the elastic response of a cantilever was investigated theoretically and numerically. The deflection of cantilever is highly affected by the non-linearity and by taking into account the non-linearity a good matching is obtainable between the predicted and measured deflections of a cantilever. As for an application of the theory bending experiment of a cantilever coated with a thin hard layer was carried out after numerical analysis. The Young's modulus of coated layer was estimated by the analysis.

7. Acknowledgement

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8. References

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